Eur. Phys. J. B **14**, 535–542 (2000)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

P-adic numbers and replica symmetry breaking

G. Parisi[1] and N. Sourlas[2,a]

[1] Dipartimento di Fisica and Sezione INFN, Università di Roma "La Sapienza", Piazzale Aldo Moro, Roma 00199, Italy
[2] Laboratoire de Physique Théorique de l'École Normale Supérieure[b], 24 rue Lhomond, 75231 Paris Cedex 05, France

Received 9 June 1999

Abstract. The p-adic formulation of replica symmetry breaking is presented. In this approach ultrametricity is a natural consequence of the basic properties of the p-adic numbers. Many properties can be simply derived in this approach and p-adic Fourier transform seems to be a promising tool.

PACS. 05.20.-y Classical statistical mechanics – 64.70.Pf Glass transitions – 75.10.Nr Spin-glass and other random models

1 Introduction

In the replica approach to disordered systems one usually introduces a matrix $Q_{a,b}$ which is the stationary point of a free energy $F[Q]$; the matrix is a zero by zero matrix, with zero elements on the diagonal [1]. Such a matrix is constructed as the $n \to 0$ limit of a normal matrix with n components.

In the mean field approach one looks for stable (or marginally stable) saddle points of the free energy. When the replica symmetry is spontaneously broken, as it happens in spin glasses, one assumes that the saddle point is given by a matrix Q constructed in a hierarchical way, which corresponds to breaking the replica symmetry group (the permutation group of n elements) in a peculiar way [2,3]. The aim of this note is to expose some hidden algebraic properties of this matrix and to show that the whole construction may be simply done using p-adic numbers.

In this approach the ultrametric properties of the matrix Q [4,5] arise naturally from the ultrametric properties of the p-adic numbers. Although we do not obtain new results in this way, we hope that this reformulation may be a useful starting point for simplify some of the long computations involved in the evaluation of the corrections to the saddle point approximation.

In Section 2 will be present the basic properties of the p-adic construction and show its equivalence to the usual hierarchical construction. In the next section the limit $n \to 0$ is performed in a simple way. In Section 4 we present an alternative and more interesting procedure for doing the limit $n \to 0$, where we connect this approach to standard p-adic analysis. In the next section we show the advantages of using the p-adic Fourier transform. Finally there are two appendices, the first dedicated to the foundations of p-adic analysis, the second to the basic properties of the Fourier transform [6,7]. Both appendices can be skipped by readers experts on p-adic analysis.

2 The p-adic construction of the matrix Q

We start the construction of the matrix Q by considering a number p (which for simplicity we suppose to be a prime number) and by assuming that $n = p^L$ for some value of L. We are going to construct the matrix Q for integer L and p in a specific way which we will discuss later. The limit $n \to 0$ will be done at the end. The matrix Q enters in the evaluation of the free energy in spin glasses and related models in the saddle point approximation. Here we do not address the point of the evaluation of the free energy and we only consider the construction of the matrix Q.

Eventually $n = p^L$ must go to zero and we can follow two options in order to realise this goal:

a) We take a value of p greater than one and we send L to minus infinity. Eventually we may do an analytic continuation in p to non integer values of p.

b) We first do an analytic continuation in p to non integer values of p. We take a value of p less than one and we send L to plus infinity.

In both cases one obtains the limit $n \to 0$. The two constructions are roughly equivalent. It seems that the second one is more simple to work with, however for pedagogical reasons we will start by presenting the first one in Section 3, while the second one will be presented in Section 4.

The first steps are common to both strategies. The construction of the matrix Q for integer p and L can be done as follows. We assume that the matrix $Q_{a,b}$

[a] e-mail: sourlas@physique.ens.fr

[b] Unité Propre du Centre National de la Recherche Scientifique, associée à l'École Normale Supérieure et à l'Université de Paris-Sud.

is of the form

$$Q_{a,b} = Q(a-b), \quad (1)$$

where $Q(k) = Q(-k)$ (symmetric matrix) and $Q(k+n) = Q(k)$. This choice restricts very much the form of the matrix and shows an explicit symmetry of this parameterization (*i.e.* translational invariance in internal space). The condition $Q(0) = 0$ implies that the elements on the diagonal are equal to zero.

The second step consists in assuming that the function $Q(k)$ is a function of the p-adic norm $|k|_p$. The appendices provide a brief introduction to peadic analysis.

In other words we suppose that

$$Q(k) = q(|k|_p). \quad (2)$$

This corresponds to setting

$$Q_{a,b} = q_i \equiv q(p^{-i}), \quad \text{if} \quad |a-b|_p = p^{-i}. \quad (3)$$

Before performing the limit $n \to 0$ it is convenient to compare our approach with the standard hierarchical construction.

In the usual case [2] one introduces a sequence of $K+2$ numbers m_i, with $m_0 = 1$ and $m_{K+1} = n$, such that m_{i-1} divides m_i for $i = 1, K+1$. One sets for $a \neq b$:

$$Q_{a,b} = q_i \quad \text{if} \quad I(a/m_i) \neq I(b/m_i)$$

and

$$I(a/m_{i+1}) = I(b/m_{i+1}), \quad (4)$$

where the function $I(z)$ is the integer part of z, *i.e.* the largest integer less or equal to z.

Let us consider the special case where the m_i are given by

$$m_i = p^i. \quad (5)$$

We want to show that the matrix Q obtained in this way coincides, after a permutation with the matrix Q constructed before with $K+1 = L$. The proof is rather simple. We associate to the index a the L digits of $a-1$ in base p:

$$a = 1 + \sum_{i=0,K} a_i p^i. \quad (6)$$

These digits form a L dimensional vector with components in the range $0 - (p-1)$. The hierarchical construction corresponds to set $Q_{a,b} = q_i$ if $a_j = b_j$ for all $j \geq i$ and $a_{i-1} \neq b_{i-1}$.

We now associate to an index a its transpose (a_T), which is obtained by writing its digits in the inverse order:

$$a_T = 1 + \sum_{i=0,K} a_i p^{(K-i)}. \quad (7)$$

The previous condition becomes that the $K-i$ less significative digits of a_T and b_T do coincide, and the $(K-i+1)$th digit differs. This last condition may be restated by saying that $a-b$ is a multiple of $p^{(K-i)}$ but not of $p^{(K-i+1)}$, *i.e.* $|a-b|_p = p^{-(K-i)}$. Apart from a reshuffling of the indices, *i.e.* a permutation, the usual construction is equivalent the p-adic construction presented before.

Generally speaking it is possible to prove that independently from the condition in equation (5), after a similar reshuffling of the indices, the hierarchical matrix Q (defined in Eq. (4)) can always be written under the form of equation (1). It is likely that many of the unexpected properties of the hierarchical construction arise from the possibility of choosing an ordering of the indices in such a way that the hierarchical matrix is invariant of under the transformation $a \to a+1$. This invariance implies that the elements of one line are the permutation of the elements of another line, however the converse is not true.

3 The n → 0 limit

We can now perform the $n \to 0$ limit. We will firstly follow the strategy a).

This limit can be reached by sending L to $-\infty$. The continuation of the usual formulae from positive to negative L can be done if we introduce the quantities q_i for non positive i.

For example let us consider the sum of the elements of a line of the matrix. In order to perform the $n \to 0$ limit we slightly change the notation of the previous section and we set

$$Q_{a,b} = q_i, \quad \text{if} \quad |a-b|_p = p^{L-i}. \quad (8)$$

We easily get that the sum is given

$$\sum_b Q_{a,b} = \sum_{i=1,L} (p-1)p^{i-1} q_i. \quad (9)$$

Indeed the number of integers k such that $|k|_p \leq p^{-j}$ (*i.e.* the volume of the p-adic sphere) is given by p^{L-j} and therefore the number of integers k such that $|k|_p = p^{-j}$ (*i.e.* the volume of the p-adic shell) is given by $(p-1)p^{L-j-1}$.

We are free to write the last equation as

$$\sum_{i=-\infty,L} (p-1)p^{i-1} q_i - \sum_{i=-\infty,0} (p-1)p^{i-1} q_i, \quad (10)$$

by introducing the extra parameters q_i for $i < 1$, which are irrelevant for positive L.

The analytic continuation to negative L can be now trivially done. In the limit $L \to -\infty$, the first term disappears and we get

$$\sum_b Q_{a,b} \to \sum_{i=-\infty,0} (p^{i-1} - p^i) q_i. \quad (11)$$

A similar procedure can be followed in order to compute other functions of the matrix Q. By comparing the previous equation with the usual ones, we see that we obtain the hierarchical formulation where a function $q(x)$ is introduced, with the extra constraint that $q(x)$ is piecewise constant with discontinuities at $x = p^{-i}$. The usual formulation, where $q(x)$ is a continuous function can be obtained by analytic continuation in p up to the point $p = 1^+$.

By performing the explicit computations similar results are obtained for the other quantities and it is possible to show that the usual approach is recovered.

4 The upsidedown world

In the other possible approach to the $n \to 0$ limit (b), we firstly do an analytic continuation in p to values less than one and only later we send $L \to +\infty$ in such a way that $p^L \to 0$. At a later stage we are free to send $p \to 1^-$ in order to reach the continuous limit. In this way we get formulae quite similar to the previous one, with the advantage that only the q_i with positive i are needed.

In this approach one obtains that the function $q(x)$ is given by

$$q(p^i) = q_i, \tag{12}$$

where the index i ranges from 0 to $+\infty$ in such a way that when $p \to 1^-$, $x \equiv p^i$ spans the interval 0-1. The formulae one obtains in this approach for $p < 1$ coincide with the formulae obtained with the formalism of the previous section (with the substitution of p with p^{-1}).

The advantage of this procedure is that we obtain formulae that are very similar to those used in the p-adic integral and that are well known to mathematicians. The strategy to prove these formulae is quite similar and therefore one can use some of the well known results in this field.

In the region where $p < 1$ it may be convenient to introduce the notation:

$$|k| = \frac{|k|_p}{p^L} \cdot \tag{13}$$

In this way $|k|$ belongs to the interval $0 - 1$, with the exception $|0| = \infty$. In the limit where $p \to 1^-$, $|k|$ spans the interval 0-1. Equation (12) can thus written as

$$Q(a-b) = q(|a-b|). \tag{14}$$

Let us apply this strategy to the computation of the sum of the elements of a line of the matrix. We find that

$$\lim_{L\to\infty} \sum_{a=1,p^L} Q(a-b) = \sum_{i=1,\infty} (p-1)p^{i-1} q_i. \tag{15}$$

For $p < 1$ the previous equation can be written as

$$\sum_{a=1,n} Q(a-b) = \sum_{i=1,\infty} (1-p)p^{i-1} q_i, \tag{16}$$

while for $p > 1$ the r.h.s. becomes proportional to the p-adic integral which is denoted as

$$\int_p \mathrm{d}a Q(a). \tag{17}$$

With some abuse of notation we denote for $p < 1$

$$\lim_{n\to 0} \sum_{a=1,n} Q(a-b) = \int_p' \mathrm{d}a Q(a), \tag{18}$$

where the sign $'$ over the integral $\int_p'$ denotes that the value zero is excluded from the integration range. We must note that the measure of the integral is normalized to -1. In a similar way we can use the notation

$$\lim_{L\to\infty} \sum_{b,c=1,p^L} F(|a-b|,|b-c|,|c-a|) = \int_p \mathrm{d}b\ \mathrm{d}a F(|a-b|,|b-c|,|c-a|). \tag{19}$$

For $p > 1$ we obtain the usual p-adic integral (apart from a normalization factor). The results for $p < 1$ can be obtained using the same steps as in Appendix B.

We can do the computation in the interesting case where the sum is restricted to all different indices. We have to compute

$$\lim_{L\to\infty} \sum_{b,c=1,p^L; a\neq b, a\neq c, b\neq c} F(|a-b|,|b-c|,|c-a|) \equiv \lim_{L\to\infty} \sum_{b,c=1,p^L}{}' F(|a-b|,|b-c|,|c-a|) \tag{20}$$

$$= \int_{p\ b,c=1,p^L; a\neq b, a\neq c, b\neq c} \mathrm{d}b\ \mathrm{d}c F(|a-b|,|b-c|,|c-a|)$$

$$= \int_p' \mathrm{d}a\ \ \mathrm{d}b F(|a-b|,|b-c|,|c-a|), \tag{21}$$

where we denote by $\sum{}'$ the sum restricted to the case of all different indices.

In the same way we could define

$$(-1)^M \lim_{n\to 0} \frac{1}{n} \sum_{a_1,a_2,\ldots a_M} F[a] \equiv \int' \mathrm{d}a_1\ \mathrm{d}a_2 \ldots \mathrm{d}a_M, F[a] \tag{22}$$

where p is less than 1 and the sum is done on all different indices. The factor $(-1)^M$ has the effect of giving a positive result for the integral. Generally speaking in this way the evaluation of sums can be reduced to the computation of quantities that are very similar to the corresponding p-adic integral.

For example let us use this strategy to compute

$$\int_p' \mathrm{d}a\ \mathrm{d}b F(|a-b|,|b-c|,|c-a|). \tag{23}$$

The application of the previous formulae tells us that the integral is given by

$$
\begin{aligned}
\sum_{b,c,b\neq a,c\neq a,b\neq c} & F(|a-b|,|b-c|,|c-a|) \\
&= \sum_{i,k,l} \mu(i,k,l) F(p^k,p^i,p^l) \\
&= \sum_{i,k;i<k} (p^{i+1}-p^i)(p^{k+1}-p^k)[F(p^k,p^k,p^i) \\
&+ F(p^k,p^i,p^k) + F(p^i,p^k,p^k)] \\
&+ \sum_i (p^{i+1}-p^i)(p^{i+1}-2p^i) F(p^i,p^i,p^i). \qquad (24)
\end{aligned}
$$

The proof can be obtained using the same strategy as in the Appendix A for computing the measure of three intersecting p-adic shells.

Finally in the continuum limit where p goes to 1^- one get the formula

$$
\begin{aligned}
&\sum_{b,c,b\neq a,c\neq a,b\neq c} F(|a-b|,|b-c|,|c-a|) = \\
&\int \mathrm{d}x\mathrm{d}y\theta(x-y)[F(x,x,y)+F(x,y,x)+F(y,x,x)] \\
&\qquad + \int \mathrm{d}x\ x\ F(x,x,x). \qquad (25)
\end{aligned}
$$

The same formula could be simply written as

$$
\begin{aligned}
&\sum_{b,c,b\neq a,c\neq a,b\neq c} F(|a-b|,|b-c|,|c-a|) = \\
&\qquad \int \mathrm{d}x\mathrm{d}y\mathrm{d}x\mu(x,y,x)F(x,y,z) \\
&\mu(x,y,z) = \theta(x-y)\delta(x-z) + \theta(x-z)\delta(x-y) \\
&\qquad + \theta(y-x)\delta(y-z) + x\delta(x-y)\delta(x-z). \qquad (26)
\end{aligned}
$$

It is important to note that the ultrametricity inequality works at reverse in the region $p<1$ and consequently also in the limit $p \to 1^-$. This is in agreement with the fact that $1-x$, not x, has the physical meaning of distance.

After some work one can find simple rules for generic sums of the type

$$
\frac{-1}{n} \sum_{a,b,c,d}{}' F(a,b,c,d) = \int_p' F(a,b,c,d) \qquad (27)
$$

where F depends only on the p-adic distance and all indices are different [9]. In the case where the function is symmetric one finds that in the limit $p \to 1^-$

$$
\begin{aligned}
&\int_p' F(a,b,c,d) = \\
&\int_{x<y<z} \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z F|_{|a-b|=|a-c|=|a-d|=x,|b-c|=|b-d|=y,|c-d|=z} \\
&\qquad + 11\ \mathit{permutations} \\
&+ \int_{x<y;x<z} \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z F|_{|a-b|=z,|b-c|=|b-d|=|a-c|=|a-d|=x,|c-d|=y} \\
&\qquad + 2\ \mathit{permutations} \\
&+ \int_{x<y} \mathrm{d}x\ \mathrm{d}y\ x\ F|_{|a-b|=y,|b-c|=|b-d|=|a-c|=|a-d|=x=|c-d|=x} \\
&\qquad + 5\ \mathit{permutations} \\
&+ \int_{x<y} \mathrm{d}x\ \mathrm{d}y\ y\ F|_{|a-b|=|b-c|=|a-c|=y,|b-d|=|a-d|=x=|c-d|=x} \\
&\qquad + 3\ \mathit{permutations} \\
&+ 2! \int \mathrm{d}x\ x^2 F(_{|a-b|=|a-c|=|a-d|=|b-c|=|b-d|=|c-d|=x}), \qquad (28)
\end{aligned}
$$

where the formula for the intersection of 4-p-adic sphere of the same radius has been crucial to obtain the last term.

If we apply the same strategy to more complicated sums we can find the formula of reference [8], where the result is written as sum over all possible trees, with a specific integral associated to a given tree.

5 Using the p-adic Fourier transform

An interesting application of this approach can be done to the formula for the product of two matrices A and B:

$$
C_{i,k} = \sum_j A_{i,j} B_{j,k}. \qquad (29)
$$

If the matrices have the form discussed here one finds that the previous formula can be written as a convolution

$$
C(i-k) = \sum_j A(i-j)B(j-k) = \sum_j a(|i-j|)b(|j-k|). \qquad (30)
$$

Finally one finds using the previous formulae that

$$
c(|i|) = \int_p \mathrm{d}k\ a(|i-k|)b(|k|), \qquad (31)
$$

which using the rules of p-adic integral after performing the limit $p \to 1^-$ can be written as:

$$
\begin{aligned}
c(x) &= \int_x^1 \mathrm{d}y(a(y)b(x)+a(x)b(y)) \\
&+ \int_x^1 \mathrm{d}y a(y)b(y) + x\ a(x)b(x). \qquad (32)
\end{aligned}
$$

Convolutions can be strongly simplified in Fourier space. In principle we can just do ordinary Fourier transform, where the momentum q is in the interval $(-\pi , \pi)$, however it is convenient to take into account the p-adic nature of the functions we consider.

We can start from the analysis leading to formula (B.8) of Appendix B. Generalizing the computations to the case where $p < 1$ and performing the continuum limit, one finds that the p-adic Fourier transform[1] of a function $A(k) = a(|k|)$ is a function $A[M] = a[y]$, defined for y in the interval 0-1, where $y = |M|^{-1}$. One must be careful in removing the factor $\frac{1}{p^L}$ in the normalization of the Fourier transform, which would be harmful here.

Equation (B.8) may be transformed to

$$A[M]|_{|M|_p=p^j} = \sum_{k=0,j} \left(p^{-k+L}(1-1/p)a_{L-k}\right) - p^{-j+L}a_{L-j} + A(0). \quad (33)$$

Let us define (for $p < 1$) the function

$$a[y] = A[M]|_{|M|_p=p^L y}. \quad (34)$$

The function $a[y]$ is defined only for y of the form p^{-k} for integer k. With this definition one finally obtains that

$$a[y] = \sum_{k=0,j} p^k(1-1/p)a(p^{-k}) - p^j a(p^{-j}) + A(0). \quad (35)$$

The final formulae in the continuum limits are

$$a[y] = a(\infty) - \int_y^1 \mathrm{d}x a(x) - y a(y),$$
$$a[\infty] = a(\infty) - \int_0^1 \mathrm{d}x a(x), \quad (36)$$

where we use the apparently strange notation $a(\infty) = A(0)$ and $a[\infty] = A[0]$.

The inverse Fourier transform formulae simply read

$$a(x) = a[\infty] - a[0] + \int_0^x \frac{\mathrm{d}y}{y}\frac{\mathrm{d}a[y]}{\mathrm{d}y},$$
$$a(\infty) = a[1]. \quad (37)$$

These relations become simpler in differential form. For example one gets:

$$x\ a'(x) = a'[x]. \quad (38)$$

This differential relation is equivalent to the integral relation in equation (36, 37) for obtaining the Fourier transform if they are complemented by the value of the Fourier transform in a given point. A possible choice is

$$a[1] = a(\infty) - a(1).$$

[1] We use the same notation as the appendix and we use the square parenthesis, $[\cdot]$, to denote the Fourier transform.

With some work one can verify that the multiplication of two matrices becomes the simple multiplication of their Fourier transform [10]:

$$c[y] = a[y]\ b[y]. \quad (39)$$

Indeed diffentiating equation (32) one finds

$$c'(y) = a'(x)(\int_x^1 \mathrm{d}y b(y) + x b'(x))$$
$$+ b'(x)(\int_x^1 \mathrm{d}y a(y) + x a'(x)). \quad (40)$$

The function $a[y]$ was already introduced in reference [10] in order to solve the inversion problem, although its p-adic nature was not recognized. Also the usual procedure of simplifying the saddle point equations by differentiating them correspond to consider the p-adic Fourier transform.

We could also consider the problem of computing the inverse of a matrix Q, *i.e.* of finding matrix R, such that

$$\sum_c Q_{a,c}R_{c,b} = \delta_{a,b}. \quad (41)$$

It is not a surprise that we find the Fourier transform of the matrix R is simply the inverse of the Fourier transform of the matrix Q:

$$q[y] = 1/r[y]. \quad (42)$$

An extremely important problem consists in the computation of the inverse of the Hessian coming from the fluctuation around the saddle point. Here one has to solve the equation

$$\sum_{e,f} M_{a,b;e,f}G_{e,f;c,d} = \delta_{a,b;c,d}. \quad (43)$$

This inversion is not a simple job and rather complex computations have been done [11,12]. However the final formulae are remarkable simple. Although we are not able at the present moment to derive these formulae in the framework of the p-adic formalism it may be useful to show that they have a very simple interpretation in terms of p-adic Fourier transform [7].

We will consider here only the so called replicon sector for which the results are simpler than in the other sectors. In fact it was shown in reference [7] that in the "longitudinal sector" one needs a generalization of the p-adic Fourier transform presented here. This generalization is presented in reference [7]. We restrict our analysis to the region where $|a-b| = z > |a-c| = x_1, |b-d| = x_2 > z$. In this region ultrametricity implies that $z = |a-d| = |b-c| = |c-d|$. Both M and G are functions of x_1, x_2, z only and we write them as $M^z(x_1, x_2)$ and $G^z(x_1, x_2)$. In the same way we denote by $G^z_{\mathrm{R}}(x_1, x_2)$ the replicon contribution to G, where the precise definition of the replicon can be found in the original papers [11,12].

Following [11] we can thus introduce the Fourier transform with respect to x_1 and x_2, which is given by

$$m^z(x_1, y_2] = m^z(x_1, \infty) - \int_{y_2}^{1} \mathrm{d}x_2 \; m^z(x_1, x_2) - y_2 m^z(x_1, y_2),$$
$$m^z[y_1, y_2] = m^z(\infty, x_2] - \int_{y_1}^{1} \mathrm{d}x_1 \; m^z(x_1, y_2] - y_1 m^z(x_1, y_2]. \quad (44)$$

One finally finds that the final formula for the replicon propagator [12] may be obtained with slightly modified inverse Fourier transform:

$$g_{\mathrm{R}}^z[x_1, x_2] = \frac{1}{m^z[x_1, x_2]}, \quad (45)$$
$$x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} g_{\mathrm{R}}^z(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \frac{1}{m^z[x_1, x_2]},$$
$$g_{\mathrm{R}}^z(z, x_2) = g_{\mathrm{R}}^z(x_1, z) = 0. \quad (46)$$

Equivalently we could write the last equation as

$$g_{\mathrm{R}}^z(x_1, x_2) = \int_{x_1}^{z} \mathrm{d}x_1 \int_{x_2}^{z} \mathrm{d}x_2 \frac{1}{x_1 x_2} \frac{\partial^2}{\partial x_1 \partial x_2} \frac{1}{m^z[x_1, x_2]} \, . \quad (47)$$

The differential relations in the inverse Fourier transform are preserved, only the second condition which fixes the value of the inverse Fourier transform in one point is modified. With these modifications the replicon sector of the inverse is just the numerical inverse of the matrix M in Fourier space.

The precise reason for the appearance of these simple formulae with a strong p-adic flavor is not completely clear at the present moment. They show the usefulness of the p-adic formalism. It would be also extremely interesting to study if the same formalism could be applied to the off equilibrium dynamics of the kind studied in reference [13].

After completion of this work we received a paper by V.A. Avetisov, A.H. Bikulov, S.V. Kozyrev [14] where some similar results are derived.

Appendix A: p-adic numbers

Let us consider a prime number p. Any integer k can be written in an unique way as

$$k = p^i \sum_{l=0}^{\infty} a_l p^l \quad (\mathrm{A.1})$$

with $i \geq 0$, $0 \leq a_l \leq p-1$ and $a_0 \neq 0$. The p-adic norm of such an integer k (*i.e.* $|k|_p$) is defined as

$$|k|_p = p^{-i}. \quad (\mathrm{A.2})$$

The p-adic norm of 0 is defined to be equal to zero. The value of the p-adic norm tells us the number of consecutive zeros at the end of a number, when it is written in base p. For a rational number $r = a/b$, the p-adic norm is defined as $|r|_p = |a|_p/|b|_p$.

The properties of the p-adic norm are well studied by mathematicians, one of the most famous property being ultrametricity, which states that

$$|a-b|_p \leq \max(|a-c|_p, |c-b|_p) \quad (\mathrm{A.3})$$

for any choice of c. This property, which generalises the statement *the sum of two even number is even*, can be proved as follows.

Using the translational invariance of the metric, we first write the ultrametric inequality in an equivalent way as

$$|a+b|_p \leq \max(|a|_p, |b|_p). \quad (\mathrm{A.4})$$

If a is a multiple of p^i (and not of p^{i+1}), and b is a multiple of p^k, with $k \geq i$, it is evident that $a+b$ is a multiple of p^i. Therefore

$$|a+b|_p \leq p^{-i} = |a|_p = \max(|a|_p, |b|_p). \quad (\mathrm{A.5})$$

We stress that we have used in a crucial way the fact that $p > 1$ for a *true* prime. (In this paper we make an analytic continuation to $p < 1$. In that case, the inequality sign would be reversed.) A direct consequence of this inequality is that any triangle is either equilateral or isosceles with the two largest sides equal. It follows that any point a inside the p-adic disk centered at o and of radius r, *i.e.* such that $|a-o|_p \leq r$, is also a center of the disk; *i.e.* if $|b-o|_p \leq r$, then also $|a-b|_p \leq r$.

The whole p-adic field may be constructed starting from the p-adic rationals by considering the closure of the rationals with respect to the p-adic norm in the same way that the real numbers (of the interval 0−1) are constructed as the closure of the rationals (of the interval $0-1$) with respect to the usual Euclidean norm.

Closing the rational field with respect to the previously defined norm one obtains the p-adic field. Continuity of a p-adic function can be defined as usual. For example a function f is continuous at the point k if

$$\lim_{n \to \infty} f(k_n) = f(k), \quad (\mathrm{A.6})$$

for any sequence of k_n which converges to k in p-adic sense (*i.e.* $|k_n - k|_p \to 0$). The extension of a function from integers to p-adic numbers is called p-adic interpolation. Here we do not need to discuss this point any more.

From our point of view a more interesting construction is the integral over the p-adic integers which can be defined in an elementary way as

$$\lim_{L \to \infty} \frac{1}{p^L} \sum_{a=1}^{p^L} F(a) = \int_p \mathrm{d}a F(a). \quad (\mathrm{A.7})$$

There are many well known properties of the p-adic integral. Here we report some of them, leaving the proof to the reader (the lazy reader can found them in any book on p-adic calculus).

a) The measure of the p-adic sphere of radius p^{-i} centered around an arbitrary point a (*i.e.* the measure of all points such that $|a-b|_p \le p^{-i}$) is given by p^{-i}. As far as the p-adic distance among integers cannot be larger than 1, the unit sphere coincides with the whole space and has measure 1.

b) The measure of the p-adic shell of radius p^{-i} centered around an arbitrary point a (*i.e.* the measure of all points such that $|a-b|_p = p^{-i}$) is given by $p^{-i} - p^{-i-1} = (1-p^{-1})p^{-i}$.

c) The measure of the intersection among two p-adic shells has rather interesting properties. Let us consider the intersection of a shell of radius p^{-i} centered around the point a with a shell of radius p^{-k} centered around the point b. The measure depends on the distance among the points a and b, which we assume to be equal to p^{-j}. Ultrametricity tell us that the measure is zero unless two among the distances coincide and the two equal distances are the largest. After some reflection one finds that only three cases have to be considered.

- We first consider the case $i = k < j$. Here the ultrametricity inequality implies that the two shells coincide and therefore the measure of the intersection is simply given by $(1-p^{-1})p^{-i}$.
- We now consider the case $i = j < k$. Here the ultrametricity inequality implies that the second shells is fully contained in the first one and therefore the measure of the intersection is simply given by $(1-p^{-1})p^{-k}$.
- We finally consider the less trivial case is $i = j = k$. If one notice that the two spheres of radius p^{-i} centered in a and b coincide and that the two spheres of radius p^{-i-1} centrex in a and b have zero intersection one finds that the measure of the intersections of the two shells is given by $(1-2p^{-1})p^{-i}$,

d) The generalization of the previous arguments allows us to compute the measure of the intersection of many p-adic shells by using ultrametricity in a systematic way. The most significative result is that the intersection of M shells of radius p^{-i}, whose centrex are all at mutual distance p^{-i} is given by $(1-Mp^{-1})p^{-i}$. The measure becomes zero for $p = M$, which implies that you cannot find $M+1$ numbers exactly at the same distance. This last results is a generalization of the well known statement that you cannot find three integers (a, b, c) such that the three differences among them $(a-b, b-c, c-a)$ are all odd.

Using the previous formulae there are a few p-adic integrals that can be obtained a simple way.

For example let us try to compute

$$\int_p \mathrm{d}a\ \mathrm{d}b F(|a-b|, |b-c|, |c-a|), \tag{A.8}$$

where for simplicity we denote by $|a|$ the p-adic norm of a.

The integral is c independent and the application of the previous formulae tells us that the integral is given by

$$\begin{aligned} &\textstyle\sum_{i,j,i\neq j} p^{-i}p^{-j}(1-p^{-1})^2 F(p^{-k}, p^{-i}, p^{-j}) \\ &\quad + \textstyle\sum_{i,j>i} p^{-i}p^{-j}(1-p^{-1})^2 F(p^{-j}, p^{-i}, p^{-i}) \qquad \text{(A.9)} \\ &\quad + \textstyle\sum_i p^{-2i}(1-2p^{-1})(1-p^{-1}) F(p^{-i}, p^{-i}, p^{-i}), \qquad \text{(A.10)} \end{aligned}$$

where $k = \min(i, j)$

Appendix B: p-adic Fourier transform

Fourier transform on the p-adic integers coincides with the usual Fourier transform. It can also be defined by analyzing the characters of the additive group. It is more simple to consider first the case where L is finite and only a finite number of points is present.

We start by considering the case in which the function $A(x)$ is defined only for $x = 1, \cdots, p^L$ (with $A(0) = A(p^L)$). The Fourier transform is defined as

$$A[M] = \frac{1}{p^L} \sum_{m=1}^{p^L} \exp(2\pi \mathrm{i} mM) A(m), \tag{B.1}$$

where M is a rational number of the form jp^{-L} with $0 \le j < p^L$. As usual the Fourier space contains the same number of points of the original space. In this paper we will use the square parenthesis to denote Fourier transform.

Let us consider the problem of computing the Fourier transform of a function which depends only on the p-adic norm,*i.e.* $A(k) = a(|k|_p)$. We are thus interested in computing

$$\begin{aligned} A[M] &= \frac{1}{p^L} \sum_{m=1}^{p^L} \exp(2\pi \mathrm{i}\ mM) \\ a(|m|_p) &= \sum_{k=0}^{L-1} a(p^{-k}) S_k[M] + \frac{1}{p^L} A(0), \end{aligned} \tag{B.2}$$

where $S_k[M]$ is the Fourier transform of the p-adic shell of radius p^{-k}:

$$S_k[M] = \frac{1}{p^L} \sum_{m=1}^{p^L} \delta_{|m|, p^{-k}} \exp(2\pi \mathrm{i}\ mM). \tag{B.3}$$

In order to compute the Fourier transform of the shell, it may be simpler to firstly compute the Fourier transform of the p-adic sphere of radius p^{-k}. A simple computation shows that

$$\begin{aligned} V_k[M] &= \frac{1}{p^L} \sum_{m=1}^{p^L} \theta(|m| - p^{-k}) \exp(2\pi \mathrm{i}\ mM) \\ &= \frac{1}{p^L} \sum_{m=1}^{p^{L-k}} \exp(2\pi \mathrm{i}\ mMp^k) \\ &= p^{-L} \exp(2\pi \mathrm{i} l p^{k-L}) \frac{\exp(2\pi \mathrm{i} l) - 1}{\exp(2\pi \mathrm{i} l p^{k-L}) - 1}, \end{aligned} \tag{B.4}$$

where we used the definition $M = lp^{-L}$.

It follows that $V_k[M] = 0$ unless $l = np^{L-k}$, $n = 0, 1, \cdots, p-1$ and the last passage is no more valid, because both numerator and denominator are equal to zero in the final result. We notice that the possible values of $|M|$ are p^j for non negative j. Consequently we find that $V_k[M] = 0$ unless $k - j \leq 0$ We also remark that, as consequence of translational invariance, the Fourier transform of a function of the p-adic norm, is still a function of the p-adic norm. We can thus define the functions $a[\ ]$ and $s_k[\ ]$ as

$$a[p^{-j}] = A[M] \mid_{|M|=p^j}$$
$$s_k[p^{-j}] = S_k[M] \mid_{|M|=p^j}. \quad \text{(B.5)}$$

The reader should notice that the functions $a[\ ]$ and $s_k[\ ]$ are defined in such a way that it argument is the range $0 - 1$. It follows that

$$v_k[p^{-j}] = p^{-k} \qquad \text{for } k - j \leq 0$$
$$v_k[p^{-j}] = 0 \qquad \text{for } k - j > 0. \quad \text{(B.6)}$$

As a consequence we find that the Fourier transform of a spherical shell is given by

$$s_k[p^{-j}] = p^{-k}(1 - 1/p) \qquad \text{for } k - j < 0,$$
$$s_k[p^{-j}] = -p^{-k} \qquad \text{for } k - j = 0,$$
$$s_k[p^{-j}] = 0 \qquad \text{for } k - j > 0. \quad \text{(B.7)}$$

The final expression for the Fourier transform thus becomes

$$a[p^{-j}] = \sum_{k=0,j} p^{-k}(1 - 1/p)a(p^{-k}) + p^{-j}a(p^{-j}) + \frac{1}{p^L}A(0). \quad \text{(B.8)}$$

It is interesting to note that the last formula the dependence on L is very simple, so that the limit $L \to \infty$ can be trivially done. Moreover most of the properties of the ordinary Fourier transform, like the theorems concerning convolutions, are still valid.

References

1. S.F. Edwards, P.W. Anderson, J. Phys. F **5**, 965 (1975).
2. For a review see M. Mézard, G. Parisi, M.A. Virasoro, *Spin glass theory and beyond* (World Scientific 1987).
3. G. Parisi, *Field Theory, Disorder and Simulations* (World Scientific 1992).
4. M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, M. Virasoro, J. Phys. France **45**, 843 (1984).
5. R. Rammal, G. Toulouse, M.A. Virasoro, Rev. Mod. Phys. **58**, 765 (1986).
6. M.H. Taibleson, *Fourier Analysis on local Fields* (Princeton University Press, Princeton, NJ, 1975).
7. C. De Dominicis, D.M. Carlucci, T. Temesvari, J. Phys. France **7**, 105 (1997); D.M. Carlucci, C. De Dominicis, *On the Replica Fourier Transform*, `cond-mat/9709200`.
8. M. Mézard, J. Yedidia (unpublished).
9. E. Marinari, G. Parisi, Phys. Lett. **203**, 52 (1988).
10. M. Mézard, G. Parisi, J. Phys. I France **2**, 2231 (1992).
11. C. De Dominicis, I. Kondor, Europhys. Lett. **2**, 617 (1986).
12. C. De Dominicis, I. Kondor, J. Phys. Lett. France **46**, L1037 (1985).
13. L.F. Cugliandolo, J. Kurchan. Phys. Rev. Lett. **71**, 1 (1993).
14. V.A. Avetisov, A.H. Bikulov, S.V. Kozyrev, `cond-mat/9904360`.